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An integrative perspective to LQ and ℓ_∞ control for delayed and quantized systems

Yorie Nakahira and Lijun Chen

***Abstract*—Deterministic and stochastic approaches to handle uncertainties may incur very different complexities in computation time and memory usage, in addition to different uncertainty models. For linear systems with delay and rate constrained communications between the observer and the controller, previous work shows that a deterministic approach, the ℓ_∞ control has low complexity but can only handle bounded disturbances. In this paper, we take a stochastic approach and propose an LQ controller that can handle arbitrarily large disturbance but has large complexity in time and space. The differences in robustness and complexity of the ℓ_∞ and LQ controllers motivate the design of a hybrid controller that interpolates between the two: The ℓ_∞ controller is applied when the disturbance is not too large (normal mode) and the LQ controller is resorted to otherwise (acute mode). We characterize the switching behavior between the normal and acute modes. Using our theoretical bounds which are supplemented by numerical experiments, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operating with low complexity most of the time.**

***Index Terms*—Robustness-complexity tradeoff, the LQ control, the ℓ_∞ control, communication constraints, robust control.**

I. INTRODUCTION

In the design of cyber-physical systems, it is essential to account for a broad range of uncertainties such as disturbances due to environmental changes and control errors due to delay and quantization in feedback loop. Two approaches are typically used to handle uncertainties: deterministic or stochastic. In the deterministic approach, uncertain input or parameters are assumed to be in an uncertainty set, and the design goal is to optimize the worst-case performance and/or make sure the system satisfies certain specifications over the uncertainty set. In the stochastic approach, uncertain input or parameter is assumed to have a certain distribution, and the design goal is usually to optimize the average performance and/or make sure the system satisfies certain specifications with high enough probability. It is obvious that the applicability of each approach depends on the characterization of uncertainty. However, it is not clear which approach incurs less complexity in time and space (i.e., memory). In the paper, we investigate some of the related issues in controller design for linear systems with delay and quantization.

Y. Nakahira is with Electrical and Computer Engineering, Carnegie Mellon University, Pittsburgh, PA 15213, USA (email: yorie@cmu.edu).

L. Chen is with Computer Science, University of Colorado, Boulder, CO 80309, USA (email: lijun.chen@colorado.edu).

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Specifically, we consider a linear dynamical system with delay and rate constrained communications between the observer and the controller; see Fig. 1 for a schematic. Previous works [2], [3] take the deterministic approach of ℓ_∞ control, i.e., to design an optimal controller that minimizes the worst-case infinity-norm of the system output under infinity-norm bounded disturbances. The resulting controller uses static memoryless quantizers and therefore has low time and space complexity. However, the efficacy of this approach partly depends on how "tight" the uncertainty set is in covering all possible disturbances, and the assumption of bounded uncertainty set will necessarily leave out disturbance that may occasionally take large values.

On the other hand, there is an extensive literature that takes the stochastic approach that can better handle (occasional) large disturbance and studies the linear-quadratic (LQ) control problem with costs (i.e., performance) in both the state and the control action; see, e.g., [4]–[8] and the related work reviewed below. Building upon controller design methods for the quantized system [1], [4], we design a controller for delayed and quantized system with a finite constant communication delay. We further derive a lower bound on the optimal performance using the method from [4]–[7], and compare the performance of the proposed LQ controller against it. The comparison shows that the LQ controller can reject large disturbance while achieving near-optimal performance. However, the LQ controller needs to store the whole distribution of the system state, which incurs a much higher time and space complexity than the optimal ℓ_∞ controller.

The above optimal/near-optimal controllers based on the two approaches have different advantages and limitations regarding robustness to uncertainty and complexity in time and space. An interesting question that arises from these differences is if it is possible to design a controller that has the advantages of both the above controllers. In this paper, we take a hybrid approach to design such a controller. Specifically, we assume that the *typical* disturbance is relatively small and covered by a bounded set, while the large disturbance (outside of the bounded set) is *rare* event that has a (tail) Gaussian distribution. Under this assumption, we construct a hybrid controller that interpolates between the ℓ_∞ controller and the LQ controller when there is no cost in the control action: The ℓ_∞ controller is applied when the disturbance is smaller than certain threshold (*normal mode*), and the LQ controller is resorted to otherwise (*acute mode*). We analyze the switching time from and the recovery time to the normal mode, as well as the performance versus complexity tradeoff. Using our theoretical bounds which are supplemented by

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numerical experiments, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operating with low complexity most of the time.

Related work and our contributions: There is a large literature on the topics studied in this paper. Here we briefly review only those that are directly relevant. Applications of the model studied in this paper range from cyber-physical systems [2], [9]–[13] to neuroscience [3] and cell biology [14], [15]. Motivated by these applications, there exists a large literature on control under communication constraints, based on either the deterministic or the stochastic approach. For the former, the stability conditions are known for a broad class of linear systems with quantization or data rate constraints [16], [17], and optimal controllers for systems with delay and quantization are given in [2], [3]. For the latter, the stability conditions are known for linear systems with quantization or data rate constraints [18]–[20], and performance bounds are given in [5]–[8], [21]–[27]. The design and performance of systems with fixed and random delays are recently investigated in [28], [29], and performance bounds are recently obtained in [30]. The relation between the optimal cost and the causal rate-distortion function is studied in [8], [31]–[38]. The information-theoretic quantities used to model communication constraints include mutual information [6], [39], anytime capacity [40], and directed information [26], [41], among others. The optimal controllers are studied for quantized systems in [4]–[7], [13], [42], [43] and references therein.

In this paper, we study the optimal controller design for *delayed and quantized* systems, and further, we take a hybrid deterministic-stochastic approach to design a hybrid controller that is robust to large disturbance and of low complexity. Our main contributions are summarized as follows:

- We characterize the optimal controller structure for the LQ system with *both delay and quantization* in communications.
- Based on the optimal controller structure, we derive a lower bound on the optimal LQ cost (Theorem 1).
- Based on the optimal controller structure, we propose a near-optimal LQ controller (Algorithm 2).
- We further propose a hybrid controller that combines the advantages of both the ℓ_∞ and LQ controllers when there is no cost in the control action (Algorithm 3), and characterize its switching behavior between the normal mode and the acute mode (Theorem 2 and Equation (25)).
- Using our theoretical bounds and supplementary numerical experiments, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, and reject occasional large disturbance while operating with low complexity most of the time.

Also, notice that we treat delay d and rate R as given, independent variables in this paper. In natural and engineering systems, however, the hardware is usually featured by certain speed-accuracy tradeoff $R = \mathcal{T}(d)$. Our results in this paper and others such as [3], combined with the speed-accuracy tradeoff, provide a theoretical foundation to systematically study the impact of the hardware speed-accuracy tradeoff on system performance; see, e.g., [44], [45] for its application in neuroscience.

Notation and preliminaries: We use lower case letter to denote a sequence, e.g., $x = \{x_0, x_1, x_2, \cdots\}$, x_τ^t to denote a truncated sequence $\{x_\tau, x_{\tau+1}, \cdots, x_t\}$ from τ to t, and for simplicity let $x^t = x_0^t$. We use $'$ to denote matrix transpose. The ℓ_∞ norm of a sequence x is defined as $\|x\|_\infty := \sup_t |x_t|$. We use $f(x)$ to denote the probability density function of a random variable x, and $f(x|y)$ to denote the conditional probability density function of a random variable x given y.

The rest of this paper is organized as follows. Section II describes the system model, as well as summarizes the existing result on the ℓ_∞ control. Section III present the LQ control design and its analysis. Section IV presents the hybrid controller and its analysis. Section V concludes the paper.

II. System Model

Consider a feedback dynamical system with delay and rate constrained communications between the observer and the controller as shown in Fig. 1. The plant follows the discrete-time dynamics:

$$x_{t+1} = Ax_t + u_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}$ is the state, $A \in \mathbb{R}$ is the dynamics constant, $w_t \in \mathbb{R}$ is the disturbance, and $u_t \in \mathbb{R}$ is the control action at time t. The system starts at time $t = 0$, and without loss of generality, we assume the initial condition $x_0 = 0$ and $w_t = 0$ for $t < 0$.

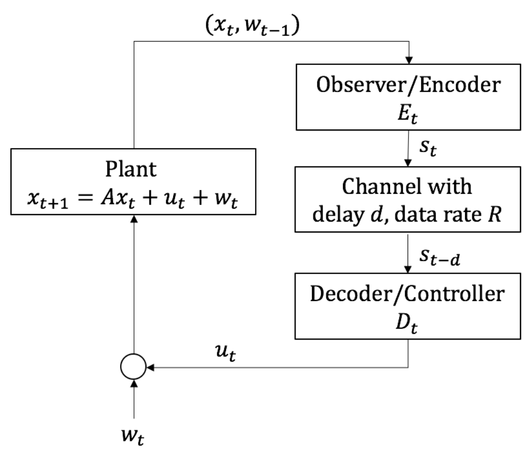

Fig. 1: The system model.

The communication channel between the observer and the controller is characterized by a finite constant delay d and a finite constant bandwidth R, with $R > \log_2 |A|$ to ensure stability [19]. Associated with the observer is an encoder that at time $t \geq 0$ is defined by a mapping E_t from the available information $\mathcal{I}_t = \{\{x_\tau\}_{\tau=0,\ldots,t}, \{w_\tau\}_{\tau=0,\ldots,t-1}\}$ to a proper codeword s_t, i.e.,

$$s_t = E_t(\mathcal{I}_t) \in S, \ t \geq 0, \tag{2}$$

where the set S of codewords has cardinality of at most 2^R. Associated with the controller is a decoder that at time $t + d$

recovers the information on state and disturbance upon the delayed information $\mathcal{J}_t = \{s_\tau\}_{\tau=0,\ldots,t}$. The controller will decide the control action u_{t+d} at time $t+d$ based on $\mathcal{J}_t$. The decoder and controller can be jointly defined by a mapping D_t:

$$u_t = \begin{cases} 0, & t < d, \\ D_t(\mathcal{J}_{t-d}), & t \geq d, \end{cases} \tag{3}$$

where no control action is taken before time d, i.e., $D_t = 0$ for $t < d$. We may loosely refer to D_t as decoder, controller or decoder-controller, whichever is more convenient in the relevant context.

Let $K := \{(E_0, D_0), (E_1, D_1), \cdots, (E_t, D_t), \cdots\}$, which we also broadly call the controller, and denote by $\mathcal{K}(R, d)$ the space of such controllers with delay d and bandwidth R. The design goal for the controller is to achieve a good performance (small state deviation under disturbance) with small control effort (small actuation, small computation time, and low memory usage), which can be quantified in terms of $\|x\|$, $\|u\|$ for certain norm $\|\cdot\|$ and by the functional form of (E_t, D_t).

A. The ℓ_∞ System

In this subsection, we summarize the existing robust control theory for the ℓ_∞ system with delay and quantization [2] [3], where the design objective is to minimize $\max_w \|x\|_\infty$. For disturbance with bounded support $\|w\|_\infty \leq L$ and stabilizing bandwidth $R > \log_2 |A|$, the optimal performance is given by:

$$\max_{\|w\|_\infty \leq L} \|x\|_\infty = \left\{ \sum_{i=0}^{d} |A^i| + \frac{|A^{d+1}|}{(2^R - |A|)^{-1}} \right\} L. \tag{4}$$

Let $\Psi(L) := \left\{|A^{d+2}|(2^R - |A|)^{-1} + |A^{d+1}|\right\} L$. The optimal performance is achieved by the ℓ_∞ controller as shown in Algorithm 1. In Algorithm 1, $\mathcal{Q}_l : \mathbb{R} \to S_R$ denotes a uniform quantizer of rate R (i.e., with $|S_R| = 2^R$ levels) over the interval $[-l, l]$. At time t, the encoder computes the quantization error q_{t+d-1} expected at time $t+d-1$. The error q_{t+d-1} plus the impact of new disturbance $A^d w_{t-1}$ is quantized/encoded, and then received at the decoder after a communication delay of d. Because of the delay, the information computed and sent at the encoder at time t is for computing the control action that will be actuated at time $t+d$. See also [46] for the extension of Algorithm 1 to the MIMO system.

Algorithm 1: The ℓ_∞ controller.

Encoder at t:
$$\begin{aligned} u^*_{t+d-1} &= -A z_{t+d-1} \\ q_{t+d-1} &= u_{t+d-1} - u^*_{t+d-1} \\ z_{t+d} &= A^d w_{t-1} + q_{t+d-1} \\ s_t &= \mathcal{Q}_{\Psi(L)/A}(z_{t+d}) \end{aligned}$$

Decoder at t:
$$\begin{aligned} u_t &= -A \mathcal{Q}^{-1}_{\Psi(L)/A}(s_{t-d}) \quad && \text{if } t \geq d \\ u_t &= 0 && \text{otherwise} \end{aligned}$$

The advantage of the ℓ_∞ controller is that it requires little computation and storage: the encoder only needs to store the last codeword and perform minimum computation, and the decoder is static and memoryless. In addition, this controller requires minimum actuation effort when $|A| \geq 1$: the stabilizing control law that minimizes $\max_{\|w\|_\infty \leq 1} \|u\|_\infty$ is identical to the above control law, which minimizes $\max_{\|w\|_\infty \leq 1} \|x\|_\infty$. However, the low complexity of the ℓ_∞ controller does not come for free. As the saturation level of the quantizer in Algorithm 1 is fixed, for a disturbance with unbounded support there is always a nonzero probability of large disturbance that makes the quantizer saturated. In such a situation, the quantization error in control can accumulate and keep increasing, rendering the system unstable. In the next section, we will consider the LQ controller that can better handle large disturbance.

III. The Linear Quadratic System

In this section, we study the robust control problem for the linear quadratic (LQ) system with delay and quantization. The disturbance w_t, $t \geq 0$ is assumed to be *i.i.d.* Gaussian with zero mean and variance σ^2, i.e., $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$. The control objective is to minimize an average cost subject to the plant dynamics (1):

$$\underset{K \in \mathcal{K}(R,d)}{\text{minimize}} \quad \lim_{N \to \infty} \frac{1}{N} \mathbb{E}\left[P x_N^2 + \sum_{t=0}^{N-1} (P x_t^2 + Q u_t^2) \right], \tag{5}$$

where $P \geq 0$ and $Q \geq 0$ balance the cost of state deviation and control action. We will first characterize the optimal controller structure and performance limit of problem (5), and then propose a near-optimal controller accordingly.

A. Optimal Controller Structure and Performance Bound

We consider an iterative process of optimizing controller for a fixed encoder, then optimizing encoder for a fixed controller, and so on [4]. To optimize controller for a fixed encoder, we first notice that, by Lemma 1 in the Appendix A, given any encoding scheme $\{E_t\}$, the optimal decoder-controller D_t for (5) has the following structure:

$$u_t = L_t \, \mathbb{E}[z_t | s^{t-d}], \tag{6}$$

where z_t is defined by the recursion

$$z_{t+1} = A z_t + A^d w_{t-d} + u_t, \quad z_0 = 0, \tag{7}$$

and

$$L_t = -(Q + P_{t+1})^{-1} P_{t+1} A \tag{8}$$

with P_t defined by the recursion

$$\begin{aligned} P_N &= P, \\ P_t &= \left[P_{t+1} + P - (Q + P_{t+1})^{-1} P_{t+1}^2\right] A^2. \end{aligned} \tag{9}$$

The optimal controller structure (6) (i.e., (28) in the Appendix A) is an extension of certainty equivalence (see [6], [22] for its definition and extension to quantized systems) to systems with delay and quantization. The auxiliary sequence $\{z_t\}$ and (6) together allow us to bound the objective value by studying an estimation problem of a Gauss-Markov source and an LQ control problem of a fully observed system. We have the following lower bound on the theoretically optimal LQ cost.

Theorem 1: The optimal performance of the robust control problem (5) is bounded below as follows:

$$\begin{aligned}&\lim_{N\to\infty}\frac{1}{N}\mathbb{E}\left[Px_N^2+\sum_{t=0}^{N-1}(Px_t^2+Qu_t^2)\right]\\ \geq\ & P\sum_{i=0}^{d-1}A^{2i}\sigma^2+P^\star A^{2d}\sigma^2+G^\star A^{2d}\frac{\sigma^2}{2^{2R}-A^2},\end{aligned}\tag{10}$$

where $P^\star$ and $G^\star$ are the unique solution to the equations:

$$\begin{aligned}P^\star&=\left[P^\star+P-(Q+P^\star)^{-1}P^{\star 2}\right]A^2,\\ G^\star&=P^\star A^2+P-P^\star.\end{aligned}\tag{11}$$

Proof: The derivation of the lower bound (10) uses the method developed in [4]–[7]. See Appendix B for the proof. ■

When there is no communication delay, i.e., $d=0$, the lower bound (10) reduces to the results of [6], [26]. The first and second terms in the lower bound, $P\sum_{i=0}^{d-1}A^{2i}\sigma^2+P^\star A^{2d}\sigma^2$, are due to delay in control action because of communication delay, while the third term $G^\star A^{2d}\frac{\sigma^2}{2^{2R}-A^2}$ is mainly due to limited data rate. Moreover, if the decoder-controller has the structure (6), the first two terms are tight in the sense that the lower bound for the cost due to delay (i.e., the first two terms) equals to the actual cost due to delay. This fact can be observed from the proof of Theorem 1.

B. The LQ Controller

Based on the optimal decoder-controller structure (6), we propose a controller, referred to as *the LQ controller*, in Algorithm 2. The encoder and decoder in Algorithm 2 use an adaptive quantizer generated by the Lloyd algorithm [4], [47], [48] and estimate z_t using recursive Bayesian estimation. The encoder computes the prior density function[1]

$$\begin{aligned}f(z_t|s^{t-d-1})&=\int_{-\infty}^{\infty}f(z_t,z_{t-1}|s^{t-d-1})dz_{t-1}\\ &=\int_{-\infty}^{\infty}f(z_t|z_{t-1},s^{t-d-1})f(z_{t-1}|s^{t-d-1})dz_{t-1},\end{aligned}\tag{12}$$

where $f(z_t|z_{t-1},s^{t-d-1})$ can be computed by

$$\begin{aligned}f(z_t|z_{t-1},s^{t-d-1})&=f(z_t|z_{t-1})\\ &=f(Az_{t-1}+A^dw_{t-d-1}+u_{t-1}|z_{t-1}).\end{aligned}$$

Then, $f(z_t|s^{t-d-1})$ is used to run the Lloyd algorithm [4], [47], [48] to find a quantizer $\mathcal{Q}_t$ that maps z_t to s_t. Given the received codeword s_{t-d} at the decoder, the update process computes the posterior density function

$$\begin{aligned}f(z_t|s^{t-d})&=\frac{f(z_t,s_{t-d}|s^{t-d-1})}{f(s_{t-d}|s^{t-d-1})}\\ &=\frac{f(z_t|s^{t-d-1})f(s_{t-d}|z_t,s^{t-d-1})}{f(s_{t-d}|s^{t-d-1})}\\ &\propto f(z_t|s^{t-d-1})f(s_{t-d}|z_t,s^{t-d-1}),\end{aligned}\tag{13}$$

[1]With a slight abuse of notation, we use $f(x|y)$ to denote both the probability density function of a random variable x conditioned on another random variable y and the function that is computed by the controller to approximate the actual density function.

where $f(z_t|s^{t-d-1})$ is the prior density function computed in (12), and $f(s_{t-d}|z_t,s^{t-d-1})$ is determined by the quantizer $\mathcal{Q}_t$. Finally, $f(z_t|s^{t-d})$ is used to generate an estimate of z_t as follows:

$$\hat{z}_t=\mathbb{E}[z_t|s^{t-d}]=\int_{-\infty}^{\infty}z_tf(z_t|s^{t-d})dz_t.\tag{14}$$

Algorithm 2: The LQ controller

Initialize:

1) Compute $f(z_d|s^0)=\mathcal{N}(0,\sigma^2)$.
2) Set $z_d=0, u_0=0$.

Encoder: At time t, the encoder performs the following procedures:

1) Update the auxiliary variable (7).
2) Generate the prior density function by (12).
3) Run the Lloyd algorithm to obtain $\mathcal{Q}_t$.
4) Send the codeword $s_t=\mathcal{Q}_t(z_t)$ to the decoder.
5) Generate the posterior density function by (13).

Decoder: At time t, the decoder receives the codeword s_{t-d} that was generated d sampling intervals before, and performs the following procedures:

1) Compute the prior density function by (12).
2) Run the Lloyd algorithm to recover $\mathcal{Q}_t$.
3) Use the delayed codeword s_{t-d} to generate the posterior density function by (13).
4) Calculate the estimate $\hat{z}_t$ of z_t by (14).
5) Compute the control action:

$$u_t=-(Q+P^*)^{-1}P^*A\,\hat{z}_t.\tag{15}$$

The proposed LQ controller may not be optimal, but can be shown to achieve near optimal performance by comparing with the lower bound (10) of the optimal performance. As mentioned in the above, the first two terms of the lower bound are tight for any delay d if the decoder-controller has the structure (6), which is the case for the proposed LQ controller. Thus, the performance gap to the lower bound reduces mostly to the difference between the achievable $G^*(z_t-\hat{z}_t)^2$ and the lower bound of $\mathbb{E}[G^*(z_t-\hat{z}_t)^2]$. Fig. 2 shows a comparison between the LQ controller and the lower bound. We see that the LQ controller achieves near optimal performance when the bandwidth R is large enough. We have also studied the performance of Algorithm 2 analytically, and the detail is presented in [42], [43].

The Gaussian distribution has infinite support, i.e., the LQ controller can handle large disturbance, as opposed to the ℓ_∞ controller that can only handle bounded disturbance. However, the LQ controller is demanding in both computation and memory, due to the use of an adaptive quantizer that is necessary for stabilizing an unstable system if the disturbance has an infinite support [17].

IV. A Hybrid Controller

We have seen from the previous sections that the ℓ_∞ controller has low time and space complexity but can only

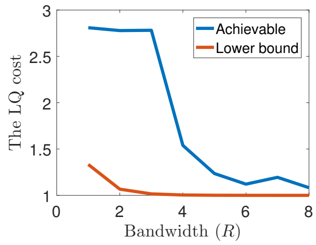

Fig. 2: The achievable performance of the LQ controller versus the lower bound (10) on the optimal performance for the system with $A=1$, $d=0$, and $\sigma^2=1$.

handle bounded disturbance, while the LQ controller can reject arbitrarily large disturbance but incurs much higher time and space complexity. An interesting question that arises from these differences is if it is possible to design a controller that has the advantages of both controllers. In this section, we take a hybrid approach to design such a controller.

Specifically, we assume that the *typical* disturbance is relatively small and covered by a bounded set, while the large disturbance (outside of the bounded set) is *rare* event that has a (tail) Gaussian distribution. Under this assumption, we construct a hybrid controller that interpolates between the ℓ_∞ controller and the LQ controller when there is no cost in the control action ($Q=0$ in problem (5)). Using analytical bounds and complementary numerical experiments, we show that the hybrid controller can achieve a sweet spot in the robustnes-complexity tradeoff, *i.e.*, reject occasional large disturbance while operate with low complexity most of the time.

A. The Hybrid Controller

In order for the ℓ_∞ and LQ controllers to be considered in a unified framework, we assume that the LQ cost function has no cost in the control action, *i.e.*, $Q=0$ in problem (5), yielding the optimal LQ controller

$$u_t = -A\hat{z}_t \tag{16}$$

to replace (15) in Algorithm 2. The proposed hybrid controller has two modes: *normal mode* that runs the ℓ_∞ controller (Algorithm 1) and *acute mode* that runs the LQ controller (Algorithm 2). We now explain the switching policy between the ℓ_∞ and LQ controllers using a bridging variable z_t and a design parameter L. Notice that the sequences $\{z_t\}$ in the ℓ_∞ and LQ controllers have identical role (storing the sum of the quantization error from past control action and the scaled disturbance $A^d w_{t-d-1}$), and thus can serve as a bridging variable to connect the two controllers. Re-define the sequence $\{q_t\}$ as

$$q_{t+1} = Aq_t + u_t + A^{d+1}w_{t-d-1} \tag{17}$$

with $w_t=0$ for $t<0$. The definition (17) does not rely on the particular realization of the controller, so q_t is well-defined in both Algorithms 1 and 2. Using q_t, z_t can be written as

$$z_t = A^d w_{t-d-1} + q_t \tag{18}$$

Algorithm 3: The hybrid controller

Initialize: $mode \leftarrow$ '$normal$'
  $\Psi(L) \leftarrow \{|A^{d+2}|(2^R-|A|)^{-1} + |A^{d+1}|\}L$
for $t\in\mathbb{N}$ **do**
  if *mode* = *'normal'* **then**
    Perform the ℓ_∞ controller (Algorithm 1)
    if $|z_t| > \Psi(L)/A$ **then**
      $mode \leftarrow$ '$acute$'
    end
  else
    Perform the LQ controller (Algorithm 2)
    if $|z_t| \le \Psi(L)/A$ **then**
      $mode \leftarrow$ '$normal$'
    end
  end
end

with the $z_t=0$ for $t\le d$. Thus, z_t in Algorithm 2 satisfies

$$\begin{aligned} z_{t+1} &= Az_t + A^d w_{t-d} + u_t \\ &= A^d w_{t-d} + Aq_t + u_t + A^{d+1}w_{t-d-1} \\ &= A^d w_{t-d} + q_{t+1}, \end{aligned} \tag{19}$$

where the first equality follows form (7), the second equality from (18), and the third equality from (17). Therefore, z_t takes the same value in both Algorithms 1 and 2. The proposed controller sets a threshold on the absolute value of z_t to determine whether the ℓ_∞ controller or the LQ controller should be used.

Let the design parameter $L\in\mathbb{R}$ be the size of the disturbance up to which the controller stays in normal mode, *i.e.*, normal mode when $\|w\|_\infty \le L$. From the derivation for its performance [3], Algorithm 1 can be shown to satisfy $|z_t| = |A^d w_{t-d-1} + q_t| \le \Psi(L)/A$ when $\|w_0^{t-d-1}\|_\infty \le L$. Conversely, $|z_t| > \Psi(L)/A$ implies $|w_\tau| > L$ for some $\tau \le t-d-1$. Thus, the condition

$$|z_t| > \Psi(L)/A \tag{20}$$

is a sufficient condition for $\|w_0^{t-d-1}\|_\infty > L$. We use this sufficient condition to define the switching policy as follows:

$$mode = \begin{cases} \text{'}normal\text{'} & |z_t| \le \Psi(L)/A, \\ \text{'}acute\text{'} & |z_t| > \Psi(L)/A. \end{cases} \tag{21}$$

The proposed hybrid controller is described in Algorithm 3.

The design parameter L impacts the system performance and controller complexity, and there exists a tradeoff between the two. We will next discuss its choice and the resulting performance and complexity tradeoff.

B. Switching Behavior

In this subsection, we analyze the behavior of the hybrid controller using the switching time from normal to acute mode and the recovery time from acute to normal mode. We denote the set of times at which the controller switches from normal to acute mode as

$$\mathcal{T}_s = \{t\in\mathbb{N} : |z_t| > \Psi(L)/A \ \ \& \ \ |z_{t-1}| \le \Psi(L)/A\},$$

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and the set of time at which the controller switches from acute to normal mode as

$$\mathcal{T}_r = \{t \in \mathbb{N} : |z_t| \le \Psi(L)/A \ \ \& \ \ |z_{t-1}| > \Psi(L)/A\}.$$

Let $t_r \in \{0\} \cup \mathcal{T}_r$ be the beginning of a normal mode, the switching time T_s is defined as

$$T_s(t_r) = \min\{t > t_r : |z_t| > \Psi(L)/A\} - t_r. \tag{22}$$

Let $t_s \in \mathcal{T}_s$ be the beginning of an acute mode, the recovery time T_r is similarly defined as

$$T_r(t_s) = \min\{t > t_s : |z_t| \le \Psi(L)/A\} - t_s. \tag{23}$$

Long switching time and short recovery time imply that the controller stays in normal mode most of the time, and thus requires less computation and memory. Therefore, the controller complexity can be roughly characterized by the time of operating in acute mode.

Let a random variable w be drawn from the same distribution with the disturbance w_t, *i.e.*, $w \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma)$. The following result characterizes the relation between the design parameter L and the expected switching time $\mathbb{E}[T_s(t_r)]$.

Theorem 2: Define a mappng $\hat{T}_s : \mathbb{R} \to \mathbb{R}_+$

$$\hat{T}_s(t_r) = \begin{cases} d + \mathbb{P}(|w| > L)^{-1} & t_r = 0, \\ \mathbb{P}(|w| > L)^{-1} & t_r \in \mathcal{T}_r. \end{cases}$$

The expected switching time $T_s(t_r)$ is lower bounded by

$$\mathbb{E}[T_s(t_r)] \ge \hat{T}_s(t_r), \tag{24}$$

and the lower bound becomes tight as the bandwidth $R \to \infty$.

Theorem 2 suggests that the expected switching time can be approximated by $\mathbb{E}[T_s(t_r)] \approx \hat{T}_s(t_r)$. To prove (24), we first decompose $\mathbb{E}[T_s(t_r)]$ into the weighted sum of the probabilities of controller switching at time $k = 1, 2, \cdots$. However, these probabilities are difficult to compute analytically. In order to circumvent this difficulty, we use tools from *majorization* (see [49], [50]) to bound their aggregate values by the weighted sum of the probabilities of disturbance w_t exceeding certain threshold, which can be easily computed. Finally, we observe that, interestingly, the weighted sum follows a *geometric distribution* which allows us to obtain semi-analytical solution. The detailed proof of Theorem 2 can be found in Appendix C.[2]

Similarly, the expected recovery time $T_r(\cdot)$ can be approximated by

$$\mathbb{E}[T_r(\cdot)] \approx \hat{T}_r = \mathbb{P}(|w| \le L)^{-1}. \tag{25}$$

Recall from (19) that the evolution of z_t follows $z_{t+1} = A^d w_{t-d} + q_{t+1}$ where q_{t+1} is a function of z_t. Assuming the quantizer (defined from the encoder and decoder) is near-optimal, a large z_{t_s} at the beginning of the an acute mode is approximately reduced by rate $|A|2^{-R}$ per unit time and by $|A^\tau|2^{-\tau R}$ after τ times. Thus, for sufficiently large $|A|2^{-R}$, the term $A^d w_{t-d}$ in (19) dominates. In this situation, observing a small disturbance, *i.e.*, $|w_{t-d}| \le L$, is enough to lessen the value of z_t below $\Psi(L)/A$. This explains why the recovery time can be approximated by a geometric distribution with success probability $\mathbb{P}(|w_t| \le L)$.

[2] To the best of our knowledge, our use of majorization to derive a semi-closed-form bound for system performance is new and not seen in the existing literature.

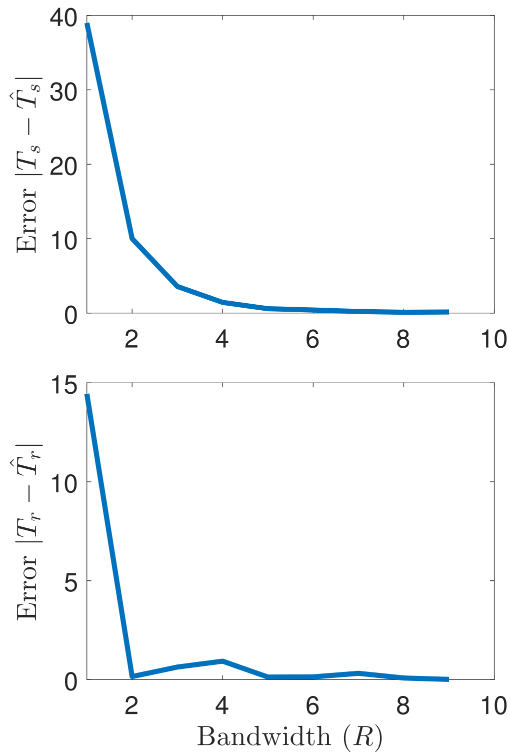

Fig. 3: The accuracy of the theoretical approximations (24) of the switching time and (25) of the recovery time for a system with $A = 1$ and $d = 1$. The empirical values of T_s and T_r are first generated by averaging 100 trials for different values of $L \in [0.1, 2]$ and $R \in \{1, 2, \cdots, 9\}$. Then, the approximation errors $|T_s - \hat{T}_s|$ and $|T_r - \hat{T}_r|$ are averaged over all L, and their mean values are plotted for different values of R.

Fig. 3 shows a comparison between the empirical value of the expected switching time $T_s(0)$ and the theoretical approximation $\hat{T}_s(0)$ and between the empirical value of the expected recovery time $T_r(\cdot)$ and the theoretical approximation $\hat{T}_r$. We see that the approximation becomes tight when the bandwidth R is large enough.

C. The Performance versus Complexity Tradeoff

The above theoretical approximations suggest that, for sufficiently large bandwidth ($|A|2^{-R} \ll 1$), a greater L implies larger switching time (from $\mathbb{E}[T_s(t_r)] \approx \hat{T}_s(t_r) = \mathbb{P}(|w_t| > L)^{-1}$) and smaller recovery time (from $\mathbb{E}[T_r(t_s)] \approx \hat{T}_r(t_s) = \mathbb{P}(|w_t| \le L)^{-1}$). This can be empirically verified; see, e.g., Fig. 4. Since the switching (recovery) time is an increasing (decreasing) function of L, the complexity of the hybrid controller decreases as L increases.

On the other hand, the decrease in controller complexity comes with cost of degraded performance because a larger L also implies a coarser quantizer in Algorithm 1 (and thus

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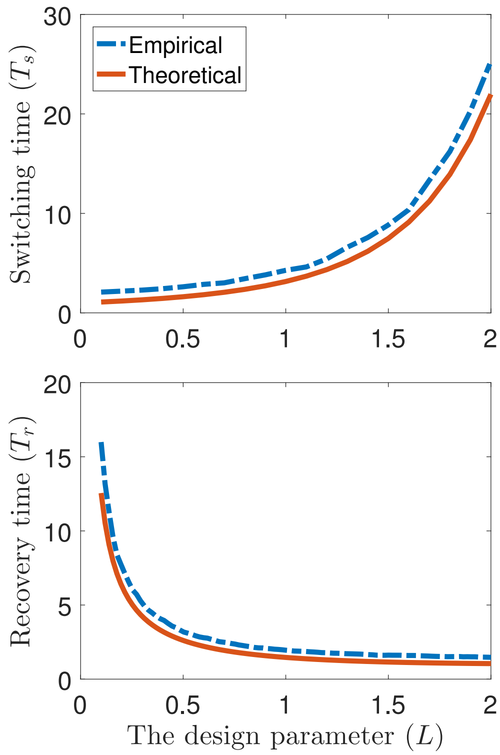

Fig. 4: The switching and recovery times as a function of L for a system with $A = 1$, $d = 1$, and $R = 6$. The averages over 100 trials are plotted for the empirical values.

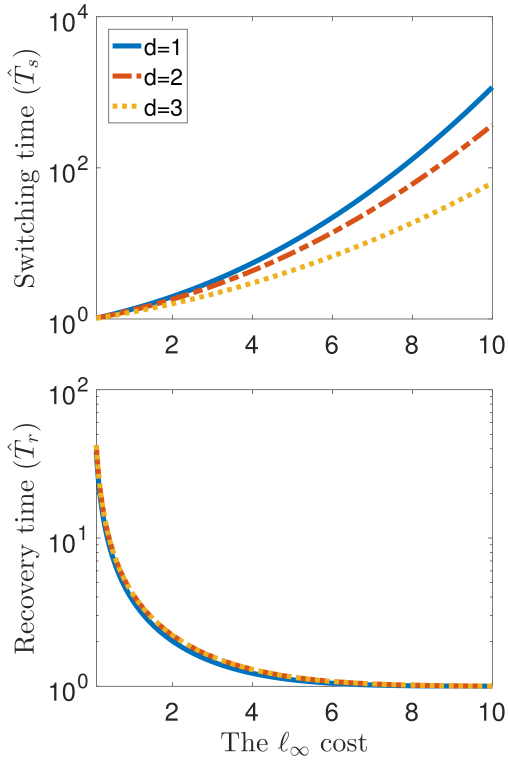

Fig. 5: The tradeoff between performance (horizontal axes) and complexity (vertical axes) for a system with $A = 1$ and $(d, R) = \{(1, 1), (2, 2), (3, 3)\}$. The system performance is quantified by the size of the error at normal mode (the ℓ_∞ cost). The complexity for running the hybrid controller increases as the switching time (duration of staying at normal mode) decreases and/or as the recovery time (time taken to return to normal mode) increases.

larger quantization error). Specifically, in normal mode,

$$|x_t| \leq \left(\sum_{i=0}^{d} |A^i| + |A^{d+1}|(2^R - |A|)^{-1} \right) L. \tag{26}$$

So, the worst-case ℓ_∞ cost in normal model is an increasing function of L, and a smaller L leads to better performance.

Therefore, there is a tradeoff between performance and complexity, as shown in, e.g., Fig. 5. Fig. 5 (and other numerical experiments) also shows that significant increase (decrease) in switching (recovery) time can be achieved with small performance degradation (notice that the vertical axes are in log-scale).

D. Performance under Mixed Disturbance

We now take a look at the performance of the proposed controllers (Algorithms 1-3) under the mixed disturbance:

$$w_t = v_t + r_t \tag{27}$$

with $v_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_v^2)$ and $\|r\|_\infty \leq 1$. We use this type of structured disturbance to model the common situation where the system experiences bounded disturbance most of the time and large disturbance occasionally (*i.e.*, with small probability).

For a feedback system with perfect communications, the optimal ℓ_∞ controller and LQ controller for the scalar system (1) are identical when the control cost is not considered. However, with communication constraints, the optimal ℓ_∞ controller and LQ controller are radically different, and the mixed disturbance poses significant challenge in encoding/decoding strategies as the system state can be defined neither in a worst-case framework nor in a stochastic framework.

The ℓ_∞ controller cannot stabilize such systems because there is a non-zero probability for the fixed quantizer to saturate. The performance of the LQ controller and the proposed hybrid controller is compared in Fig. 6. The LQ controller has degraded performance when there exists an additional disturbance r that cannot be well-defined using probability density function. However, the proposed hybrid controller consistently achieves robust performance under such disturbance. By exploiting the additional dimension in the controller design space, the right inegration of stochastic (LQ) and worst-case (ℓ_∞) enables a robust controller under communication constraints.

V. Conclussion

We have considered robust control design for linear systems with delayed and rate constrained communications between the observer and the controller. We first take a stochastic approach and propose an LQ controller that can handle arbitrarily large disturbance but has large complexity in time and space.

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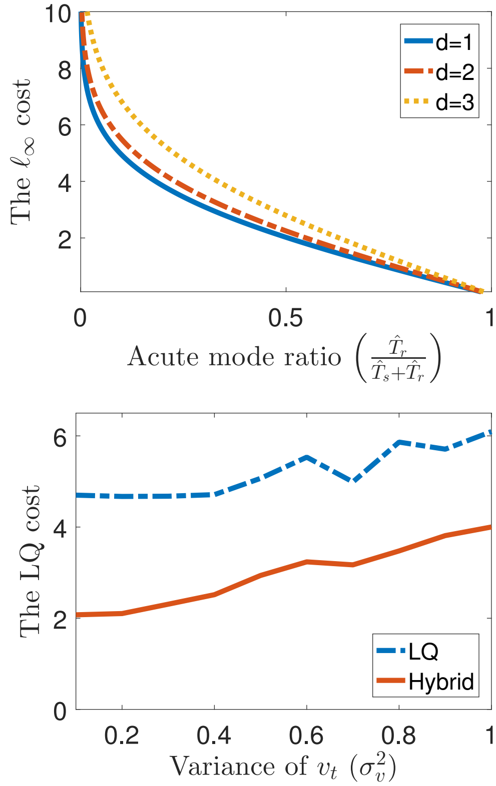

Fig. 6: Performance of the hybrid controller. The figure on the top shows the tradeoff between the normal mode performance (in the ℓ_∞ cost) and the approximated acute mode ratio $\hat{T}_r/(\hat{T}_r + \hat{T}_s)$ for a system with $A = 1$ and $(d, R) = \{(1, 1), (2, 2), (3, 3)\}$. The figure on the bottom shows the performance (in the LQ cost) for a system with $A = 1$, $d = 1$, $R = 3$ and under the mixed disturbance with different variances σ_v^2. The averaged LQ costs for 100 trials are plotted.

This is different from the ℓ_∞ control (a deterministic approach) that previous work have shown to have low time/space complexity but can only handle bounded disturbance. The differences in robustness and complexity of the LQ and ℓ_∞ controllers motivate the design of a hybrid controller that interpolates between the two: The ℓ_∞ controller is applied when the disturbance is not too large (normal mode) and the LQ controller is resorted to otherwise (acute mode). We have characterized the switching time from and the recovery time to the normal mode. Our theoretical bounds and supplementary numerical experiments show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operating with low complexity most of the time.

VI. ACKNOWLEDGMENTS

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VII. APPENDIX

In this section, we provide the proofs for the main results in the paper.

A. Lemma 1

The following lemma characterizes the structure of the optimal controller for problem (5).

Lemma 1: Consider system (1) and robust control problem (5). Given any fixed encoding scheme $\{E_t\}$, the optimal decoder-controller D_t has the following structure:

$$u_t = L_t\, \mathbb{E}[z_t | s^{t-d}], \tag{28}$$

where z_t is defined by the recursion

$$z_{t+1} = Az_t + A^d w_{t-d} + u_t, \quad z_0 = 0, \tag{29}$$

and

$$L_t = -\frac{P_{t+1}A}{Q + P_{t+1}}, \tag{30}$$

with P_t defined by the recursion

$$\begin{aligned} &P_N = P, \\ &P_t = \left(P_{t+1} + P - \frac{P_{t+1}^2}{Q + P_{t+1}}\right) A^2. \end{aligned} \tag{31}$$

We first present a result that will be used in the proof of Lemma 1. Define

$$e_t = w_{t-1} + Aw_{t-2} + \cdots + A^{d-1}w_{t-d}, \tag{32}$$
$$z_t = x_t - e_t, \tag{33}$$

where e_t captures the component in the state x_t that results from the disturbance w_{t-d}^{t-1} and cannot be mitigated due to the delay in control, while z_t depends on the information of w_0^{t-d-1} and the control action in response to it. Obviously, z_t and e_t are *independent*. Moreover, $\mathbb{E}[e_t] = 0$, and z_t satisfies equation (29), restated below

$$z_{t+1} = Az_t + A^d w_{t-d} + u_t, \quad z_0 = 0.$$

In order to decompose the effects of control action and disturbance, we define $\bar{z}_t$ to be the state z_t that would be generated at time t when the system (1) has zero control $u_t \equiv 0$. Setting $u_t = 0$ in the above equation, we obtain

$$\bar{z}_{t+1} = A\bar{z}_t + A^d w_{t-d}, \quad \bar{z}_0 = 0. \tag{34}$$

Recall that $\{s_t\}$ is the codewords generated by $\{z_t\}$. We introduce an auxiliary encoder

$$f(\bar{s}_{t-d} | \bar{z}^t, \bar{s}^{t-d-1}) = f(s_{t-d} | \bar{z}^t, s^{t-d-1}) \tag{35}$$

to generate another sequence of codewords $\{\bar{s}_t\}$.

Lemma 2: The following relation holds:

$$z_t - \mathbb{E}[z_t | s^{t-d}] = \bar{z}_t - \mathbb{E}[\bar{z}_t | \bar{s}^{t-d}].$$

Proof: (Lemma 2) We first use mathematical induction to show

$$f(s^{t-d}, \bar{z}^t) = f(\bar{s}^{t-d}, \bar{z}^t). \tag{36}$$

Obviously, (36) holds at $t = 0$. If (36) holds until t, then (36) also holds for $t+1$ because

$$\begin{aligned}&f(\bar{s}^{t-d+1}, \bar{z}^{t+1})\\ =&f(\bar{s}^{t-d}, \bar{z}^{t})f(\bar{z}_{t+1}|\bar{s}^{t-d}, \bar{z}^{t})f(\bar{s}_{t-d+1}|\bar{s}^{t-d}, \bar{z}^{t+1})\\ =&f(s^{t-d}, \bar{z}^{t})f(\bar{z}_{t+1}|s^{t-d}, \bar{z}^{t})f(s_{t-d+1}|s^{t-d}, \bar{z}^{t+1})\\ =&f(s^{t-d+1}, \bar{z}^{t+1}),\end{aligned}$$

where the second equality is due to construction (35), the induction hypothesis (36), and the fact that $f(\bar{z}_{t+1}|\bar{s}^{t-d}, \bar{z}^{t}) = f(\bar{z}_{t+1}|\bar{z}^{t}) = f(\bar{z}_{t+1}|s^{t-d}, \bar{z}^{t})$. By (36), we obtain

$$\begin{aligned}\mathbb{E}[z_t|s^{t-d}] &= \mathbb{E}\left[\bar{z}_t + \sum_{k=1}^{t} A^{k-1}u_{t-k}\Big|s^{t-d}\right]\\ &= \mathbb{E}[\bar{z}_t|s^{t-d}] + \sum_{k=1}^{t} A^{k-1}u_{t-k}\\ &= \mathbb{E}[\bar{z}_t|\bar{s}^{t-d}] + \sum_{k=1}^{t} A^{k-1}u_{t-k},\end{aligned}$$

and thus

$$\begin{aligned}&z_t - \mathbb{E}[z_t|s^{t-d}]\\ =&\bar{z}_t + \sum_{k=1}^{t} A^{k-1}u_{t-k} - \left(\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}] + \sum_{k=1}^{t} A^{k-1}u_{t-k}\right)\\ =&\bar{z}_t - \mathbb{E}[\bar{z}_t|\bar{s}^{t-d}].\end{aligned}$$

■

Lemma 2 implies that we can negate all the effect of the control action to obtain $\bar{z}_t$. Intuitively, this is because u_0^t is generated from s^{t-d}. This separation allows us to prove Lemma 1.

Proof: (Lemma 1) Consider the cost-to-go:

$$J_t(s^{t-d}) = \mathbb{E}\left[Px_N^2 + \sum_{\tau=t}^{N-1} Px_\tau^2 + Qu_\tau^2\Big|s^{t-d}\right] \tag{37}$$

for any $k < N$ and $J_N = \mathbb{E}[Px_N^2]$. We use mathematical induction to show the following properties:

(i) The optimal cost-to-go satisfies

$$J_t(s^{t-d}) = \mathbb{E}\left[P_t\hat{z}_t^2|s^{t-d}\right] + \alpha_t(s^{t-d}), \tag{38}$$

where $\hat{z}_t = \mathbb{E}[z_t|s^{t-d}]$ and $\alpha_t(s^{t-d})$ is a function of s^{t-d} whose expected value does not depend on the choice of control action, *i.e.*,

$$\mathbb{E}\left[\alpha_t(s^{t-d})\right] = \mathbb{E}\left[\alpha_t(\bar{s}^{t-d})\right]. \tag{39}$$

(ii) The optimal controller admits the form (28).

At $t = N$, the cost-to-go satisfies

$$\begin{aligned}J_N &= \mathbb{E}[Px_N^2|s^{N-d}]\\ &= \mathbb{E}[P(\hat{z}_N + \tilde{z}_N + e_N)^2|s^{N-d}]\\ &= \mathbb{E}[P\hat{z}_N^2|s^{N-d}] + \mathbb{E}[P\tilde{z}_N^2|s^{N-d}] + \mathbb{E}[Pe_N^2],\end{aligned}$$

where $\tilde{z}_t := z_t - \hat{z}_t$, and the last equality holds because e_N, $\hat{z}_N$ and $\tilde{z}_N$ are uncorrelated and e_N is independent of s^{N-d}. By Lemma 2, $\mathbb{E}[P\tilde{z}_N^2|s^{N-d}]$ does not depend on the choice of control action. Letting $\alpha_N = \mathbb{E}[P\tilde{z}_N^2|s^{N-d}] + \mathbb{E}[P_N e_N^2]$ yields (38) for $t = N$.

Assume now that (38) holds for $t = k+1$. The optimal cost-to-go at time $t = k$ can be derived as follows:

$$\begin{aligned}&J_k(s^{k-d})\\ &= \min_{u_k}\mathbb{E}[Px_k^2 + Qu_k^2 + J_{k+1}|s^{k-d}] \qquad (40)\\ &= \min_{u_k}\mathbb{E}[Px_k^2 + Qu_k^2\\ &\qquad + \mathbb{E}\left[P_{k+1}\hat{z}_{k+1}^2|s^{k-d+1}\right] + \alpha_{k+1}|s^{k-d}]\\ &= \min_{u_k}\mathbb{E}[(P + P_{k+1}A^2)\hat{z}_k^2 + (Q + P_{k+1})u_k^2\\ &\qquad + P_{k+1}Au_k\hat{z}_k + AP_{k+1}\hat{z}_k u_k|s^{k-d}] \qquad (41)\\ &\qquad + \mathbb{E}[Pe_k^2 + P_{k+1}\hat{w}_k^2 + P\tilde{z}_k^2|s^{k-d}]\\ &\qquad + \mathbb{E}[\alpha_{k+1}(s^{k-d+1})|s^{k-d}],\end{aligned}$$

where $\hat{w}_k = \mathbb{E}[A^d w_{k-d} + A\tilde{z}_k|s^{k-d+1}]$, and by induction hypothesis the second equality holds. By Lemma 2 and induction hypothesis, $Pe_k^2 + P_{k+1}\hat{w}_k^2 + P\tilde{z}_k^2$ does not depend on the control action u_t. Therefore, we can just consider minimizing the first term (41). The control action that minimizes this term is given by (28), i.e.,

$$u_k = -\frac{P_{k+1}A}{Q + P_{k+1}}\ \hat{z}_k,$$

where

$$P_k = \left(P_{k+1} + P - \frac{P_{k+1}^2}{Q + P_{k+1}}\right)A^2.$$

Substituting this control action u_k into J_k, we obtain the optimal cost-to-go

$$J_k(s^{k-d}) = \mathbb{E}\left[P_k\hat{z}_k^2|s^{k-d}\right] + \alpha_k(s^{k-d})$$

with

$$\begin{aligned}&\alpha_k(s^{k-d}) = \mathbb{E}[Pe_k^2 + P_{k+1}\hat{w}_k^2 + P\tilde{z}_k^2 + \alpha_{k+1}|s^{k-d}]\\ &= \mathbb{E}[Pe_k^2 + P_{k+1}A^2\tilde{z}_t^2 + P_{k+1}A^{2d}w_{t-d}^2\\ &\quad - P_{k+1}\tilde{z}_{t+1}^2 + P\tilde{z}_k^2 + \alpha_{k+1}(s^{k-d+1})|s^{k-d}], \qquad (42)\end{aligned}$$

where the second equality is obtained as follows. Given s^{k-d}, the random variable $\hat{w}_k$ is the estimate of $A^d w_{k-d} + A\tilde{z}_k$ given s_{k-d+1}, and the random variable $\tilde{z}_{k+1}$ is the resulting estimation error, *i.e.*,

$$\hat{w}_k + \tilde{z}'_{k+1} = A^d w_{k-d} + A\tilde{z}_k. \tag{43}$$

Therefore, the weighted covariance of the estimation target equals the sum of the weighted estimation error covariance and the weighted estimate covariance

$$\begin{aligned}&\mathbb{E}[P_{k+1}(A^d w_{k-d} + A\tilde{z}_k)^2|s^{k-d}]\\ &= \mathbb{E}[\tilde{z}_{k+1}^2|s^{k-d}] + \mathbb{E}[P_{k+1}\hat{w}_k^2|s^{k-d}]\end{aligned}$$

Combining above with

$$\begin{aligned}&\mathbb{E}[P_{k+1}(A^d w_{k-d} + A\tilde{z}_k)^2|s^{k-d}]\\ &= \mathbb{E}[P_{k+1}A^2\tilde{z}_t^2 + P_{k+1}A^{2d}w_{k-d}^2|s^{k-d}]\end{aligned}$$

yields (42). By Lemma 2 and the induction hypothesis $\mathbb{E}[\alpha_{k+1}|s^{k-d}] = \mathbb{E}[\alpha_{k+1}|\bar{s}^{k-d}]$, α_k does not depend on the choice of control action. So, equation (38) holds for $t = k$. ■

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From the proof of Lemma 1, we can observe that, given any encoder, the optimal decoder are essentially the optimal LQ controller for the sequence $\hat{z}_t$, which evolves according to the dynamics

$$\hat{z}_{t+1} = A\hat{z}_t + u_t + \hat{w}_t. \tag{44}$$

In other words, the optimal decoder are the certainty equivalent controller for the sequence z_t, the estimation target of $\hat{z}_t$. When there is no delay in the control action, *i.e.*, $d = 0$, then this optimal decoder reduces to the certainty equivalent controller for x_t, as is given by [6].

B. Proof of Theorem 1

We first describe a result that will be used later.

Lemma 3 ([6], [8]): Consider a scalar Gauss-Markov sequence $\{y_t\}$ satisfying

$$y_{t+1} = Ay_t + v_t, \quad y_0 = 0, \tag{45}$$

where $A \in \mathbb{R}$, $y_t \in \mathbb{R}$, and $v_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$. Assume that at each time t, only $R(> \log_2 |A|)$ bits of information about y^t can be transmitted to $s_t \in S$, where $|S| = 2^R$ and s_t is a function of (y^t, s^{t-1}). Let $\hat{y}_t$ be an estimate of y_t using only the information of s^t. Then, the following inequality holds:

$$\lim_{t\to\infty} \frac{1}{N}\mathbb{E}\left[\sum_{t=1}^{N}(y_t - \hat{y}_t)^2\right] \geq \frac{\sigma^2}{2^{2R} - A^2}.$$

With Lemmas 1 and 3, we are ready to prove Theorem 1.

Proof: (Theorem 1) By equation (38),[3]

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}[Px_t^2 + Qu_t^2] \\
&= \lim_{N\to\infty} \frac{1}{N}\,\mathbb{E}\left[Px_N^2 + \sum_{t=0}^{N-1} Px_t^2 + Qu_t^2\right] \\
&= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}[J_1] \\
&= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}[J_d(s^0)] \\
&= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\mathbb{E}\left[P_d\hat{z}_d^2|s^0\right] + \alpha_d(s^0)\right] \\
&= \lim_{N\to\infty} \frac{1}{N}\mathbb{E}\left[\alpha_d(s^0)\right]
\end{aligned}$$

Next we observe that $\mathbb{E}[\alpha_t]$ satisfies the relation

$$\mathbb{E}[\alpha_k(s^{k-d})] \tag{46}$$

$$\begin{aligned}
=&\mathbb{E}[\mathbb{E}[\alpha_{k+1}(s^{k-d+1}) + Pe_k^2 + P_{k+1}A^{2d}w_{k-d}^2 \\
&+ (P_{k+1}A^2 + P)\tilde{z}_k^2 - P_{k+1}\tilde{z}_{k+1}^2|s^{k-d}]]
\end{aligned} \tag{47}$$

$$=\mathbb{E}[\alpha_{k+1}(s^{k-d+1})] + \mathbb{E}[Pe_k^2 + P_{k+1}A^{2d}w_{k-d}^2 \tag{48}$$

$$+ (P_{k+1}A^2 + P)\tilde{z}_k^2 - P_{k+1}\tilde{z}_{k+1}^2] \tag{49}$$

$$=\mathbb{E}[\alpha_N(s^{N-d})] + \sum_{\tau=k}^{N-1}\mathbb{E}\Big[Pe_\tau^2 + P_{\tau+1}A^{2d}w_{\tau-d}^2 \tag{50}$$

$$+ (P_{\tau+1}A^2 + P)\tilde{z}_\tau^2 - P_{\tau+1}\tilde{z}_{\tau+1}^2\Big] \tag{51}$$

[3]With a slight abuse of notation, we use J_1 without the conditioning of the sequence s_t because it is purely determined from the initial condition.

Because the system is controllable, the Riccati difference equation (11) has a unique solution $P^\star$, and $\lim_{N\to\infty} P_N = P^\star$. Therefore, we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{\tau=d}^{N-1}\mathbb{E}[P_{\tau+1}A^{2d}w_{\tau-d}^2] = P^\star A^{2d}\sigma^2 \tag{52}$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{\tau=d}^{N-1}\mathbb{E}[(P_{k+1}A^2 + P - P_{k+1})\tilde{z}_t^2] \tag{53}$$

$$= \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N-1}\mathbb{E}[(P^\star A^2 + P - P^\star)\tilde{z}_t^2]. \tag{54}$$

Combining (46)–(50) and (52)–(54), we obtain that

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\mathbb{E}[\alpha_d(s^0)] \\
&= P(1 + A^2 + A^4 + \cdots + A^{2(d-1)})\sigma^2 + P^\star A^{2d}\sigma^2 \\
&\quad + \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N-1}\mathbb{E}[(P_{k+1}A^2 + P - P_{k+1})\tilde{z}_t^2] \\
&= P\sum_{i=0}^{d-1}A^{2i}\sigma^2 + P^\star A^{2d}\sigma^2 \\
&\quad + \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N-1}\mathbb{E}[(P^\star A^2 + P - P^\star)\tilde{z}_t^2].
\end{aligned}$$

When $x_t \in \mathbb{R}$, from Lemma 3, the second term is lower bounded by

$$\mathbb{E}[G^\star(\bar{z}_t - \mathbb{E}[\bar{z}_t|\bar{s}^{t-d}])^2] \geq G^\star A^{2d}\frac{\sigma^2}{2^{2R} - A^2}$$

Therefore, we have obtained (1). ∎

C. Proof of Theorem 2

Proof: (Theorem 2) We first prove the lower bound for $t_r = 0$. Let $\{E_k\}$ be the event that the controller switches at time k, *i.e.*,

$$E_k = \{|z_t| \leq \Psi(L)/A \text{ for all } t < k \text{ and } |z_k| > \Psi(L)/A\}.$$

Notice that $\{E_k\}$ a sequence of a mutually exclusive set of events, and that $\mathbb{P}(E_k) = 0$ for $k \leq d$ (since $z_t = 0$ for $t \leq d$ by definition). Let $\{F_k\}$ be the event that the disturbance first exceeds L in amplitude at time k, *i.e.*,

$$F_k = \{|w_t| \leq L \text{ for all } t < k \text{ and } |w_k| > L\}.$$

The sequence $\{E_k\}$ is a mutually exclusive set of events, and $\lim_{\tau\to\infty}\sum_{i=0}^{\tau}\mathbb{P}(E_i) = 1$. Same holds for $\{F_k\}$, *i.e.*, $\lim_{\tau\to\infty}\sum_{i=0}^{\tau}\mathbb{P}(F_i) = 1$. From $\cup_{i\geq k}E_i \subset \cup_{i\geq k}F_i$, we obtain

$$\sum_{i=k-d-1}^{\infty}\mathbb{P}(F_i) \leq \sum_{i=k}^{\infty}\mathbb{P}(E_i) \tag{55}$$

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for any $k \in \mathbb{N}$. Using (55), the expected switching time can be bounded below by

$$\begin{aligned}
\mathbb{E}[T_s(0)] &= \sum_{k=0}^{\infty} k\, \mathbb{P}(E_k) \\
&= \sum_{k=0}^{\infty} k\, \mathbb{P}(E_k) - \sum_{k=0}^{\infty} k\, \mathbb{P}(E_{k+d}) + \sum_{k=0}^{\infty} k\, \mathbb{P}(E_{k+d}) \\
&= d + \sum_{k=0}^{\infty} k\, \mathbb{P}(E_{k+d}) \\
&= d + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) \\
&\geq d + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1}) \\
&= d + \sum_{k=1}^{\infty} k\, \mathbb{P}(F_{k-1}) \\
&= d + \sum_{k=1}^{\infty} k\, \big(1 - \mathbb{P}(|w| > L)\big)^{k-1} \mathbb{P}(|w| > L) \\
&= d + \mathbb{P}(|w| > L)^{-1},
\end{aligned}$$

where the last equality can be interpreted as computing the mean of a geometric distribution with failure probability $\mathbb{P}(|w| > L)$.

Next, notice that $|z_t| \leq \Psi(L)/A$ and $|w_{t-d}| \leq L$ implies $|z_{t+1}| \leq \Psi(L)/A$. Thus, by the same argument, we obtain the lower bound for $t_r \in \mathcal{T}_r$:

$$\mathbb{E}[T_s(t_r)] \geq \mathbb{P}(|w| > L)^{-1}.$$

Next, we prove the convergence for $t_r = 0$, *i.e.*, $\mathbb{E}[T_s(0)] \overset{R\to\infty}{\to} d + \mathbb{P}(|w| > L)^{-1}$. Since $d + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) \geq d + \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1})$ is the only inequality from the above analysis, it is suffice to show that $|\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) - \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1})| \to 0$. By $\|q\|_\infty \overset{R\to\infty}{\to} 0$ and $z_t \to A^d w_{t-d-1}$, $\mathbb{P}(F_{t-d-1}) \to \mathbb{P}(E_t)$. This implies that

$$\begin{aligned}
&\left| \sum_{i=k-1}^{\infty} \mathbb{P}(F_{i-1}) - \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) \right| \\
&= \left| \left(1 - \sum_{i=0}^{k-2} \mathbb{P}(F_i)\right) - \left(1 - \sum_{i=0}^{k-1} \mathbb{P}(E_{i+d})\right) \right| \\
&\to 0 \text{ as } R \to \infty
\end{aligned}$$

holds for any $k \in \mathbb{N}$. Since both $\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d})$ and $\sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1})$ are bounded, for any $\epsilon > 0$ there exits a sufficiently large T such that we have for any $\tau > T$,

$$\sum_{k=\tau}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) \leq \epsilon/4 \quad \text{and} \quad \sum_{k=\tau}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1}) \leq \epsilon/4,$$

and sufficiently large $\bar{R}$ such that $R > \bar{R}$ implies

$$\sum_{k=1}^{\tau} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) \leq \epsilon/4 \quad \text{and} \quad \sum_{k=1}^{\tau} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1}) \leq \epsilon/4,$$

which jointly yields

$$\left| \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(E_{i+d}) - \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} \mathbb{P}(F_{i-1}) \right| \leq \epsilon. \tag{56}$$

The case for $t_r \in \mathcal{T}_r$ follows the same argument and is omitted here. ■

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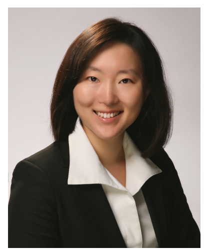
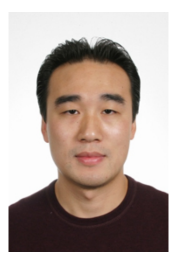


Yorie Nakahira is an Assistant Professor of Electrical and Computer Engineering at Carnegie Mellon University. She received a Ph.D. in Control and Dynamical Systems from California Institute of Technology in 2019, and a Bachelor's degree from Tokyo Institute of Technology in 2012. Her research interests include optimization, control, and information theory with applications in neuroscience and biology.



Lijun Chen (M'05) is an Assistant Professor of Computer Science at University of Colorado at Boulder. He received a Ph.D. in Control and Dynamical Systems from California Institute of Technology in 2007. He was a co-recipient of the Best Paper Award at the IEEE International Conference on Mobile Ad-hoc and Sensor Systems (MASS) in 2007. His current research interests include optimization and control of networked systems and autonomous systems, distributed optimization and control, machine learning and its integration with control, and game theory and its engineering application. He is an editor for the IEEE Transactions on Communications and editor for the IEEE Control Systems Society Conference Editorial Board. He serves as the chair of the Special Interest Group in ICT-Enabled Transactive Energy and Grid Economics of IEEE ComSoc Smart Grid Communications Emerging Technical Subcommittee 2017-2019.